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EQUIVALENT LUMPED PARAMETERS FOR A BEND IN A TWO-WIRE TRANSMISS--ETC(U)

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EQUIVALENT LUMPED PARAMETERS FOR A BEND IN A TWO-WIRE TRANSMISSION LINE

Science Applications Inc.
Berkeley, CA 94701

July 1977

Final Report

Approved for public release; distribution unlimited.

AIR FORCE WEAPONS LABORATORY
Air Force Systems Command
Kirtland Air Force Base, NM 87117

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This report calculates the equivalent lumped parameters of a bend in a parallel-wire transmission line. The equivalent lumped inductance of the bend is calculated exactly within the thin-wire assumption. The inductance for an abrupt bend is obtained explicitly in simple closed form. The inductance for a smooth bend modeled by a circular arc is reduced to one-dimensional integrals that can readily be computed. The equivalent lumped capacitance of the bend is calculated from a variational principle. The capacitance of an abrupt bend is obtained (Con't)

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CONT

20. ABSTRACT (Con't)

explicitly in simple form. The capacitance of a circular bend is expressed in terms of one-dimensional integrals to be computed numerically.

END

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I. INTRODUCTION

Electromagnetic waves propagating along a transmission line are scattered by bends in the line. The amount of scattering depends on the shapes and sizes of the bends. At sufficiently low frequencies, when the wavelengths greatly exceed the bend dimensions, the scattering effect of each bend can be represented by equivalent lumped network elements loaded onto the transmission line at the location of the bend.

Figure 1 shows a bend in an otherwise straight parallel-wire transmission line and its representation by an equivalent symmetrical T section network. The bend is formed when the two parallel wires are deflected identically through an angle α. More precisely, the vertical plane in Figure 1 defined by the two parallel wires on one side of the bend intersects the vertical plane defined by the wires on the other side at an angle $\pi - \alpha$; the line of intersection is normal to the two parallel horizontal planes containing each of the wires individually.

The objective of the present effort is to determine the lumped inductance L_d and the capacitance C_d appearing in the equivalent circuit representation of the bend in Figure 1. These lumped elements can be calculated from a pair of quasi-electrostatic and quasi-magnetostatic boundary-value problems for the bend geometry.

The analysis of the bent two-wire transmission line is relevant to the EMP internal-coupling problem of an aircraft. A cable running parallel to a metallic wall or floor in the aircraft's interior is essentially a two-wire transmission line on account of the electrical image. A quantitative knowledge of the bend inductance and capacitance will enable one to estimate the effect of a bend on the EMP propagation characteristics of the cable.

The bent parallel-wire transmission line has previously been studied by Tomiyasu [1] and King [2]. These authors limited their investigations to the abrupt V-shaped bend. The present work improves on their results and, at the same time, extends their analysis to the more general case of a smooth gradual

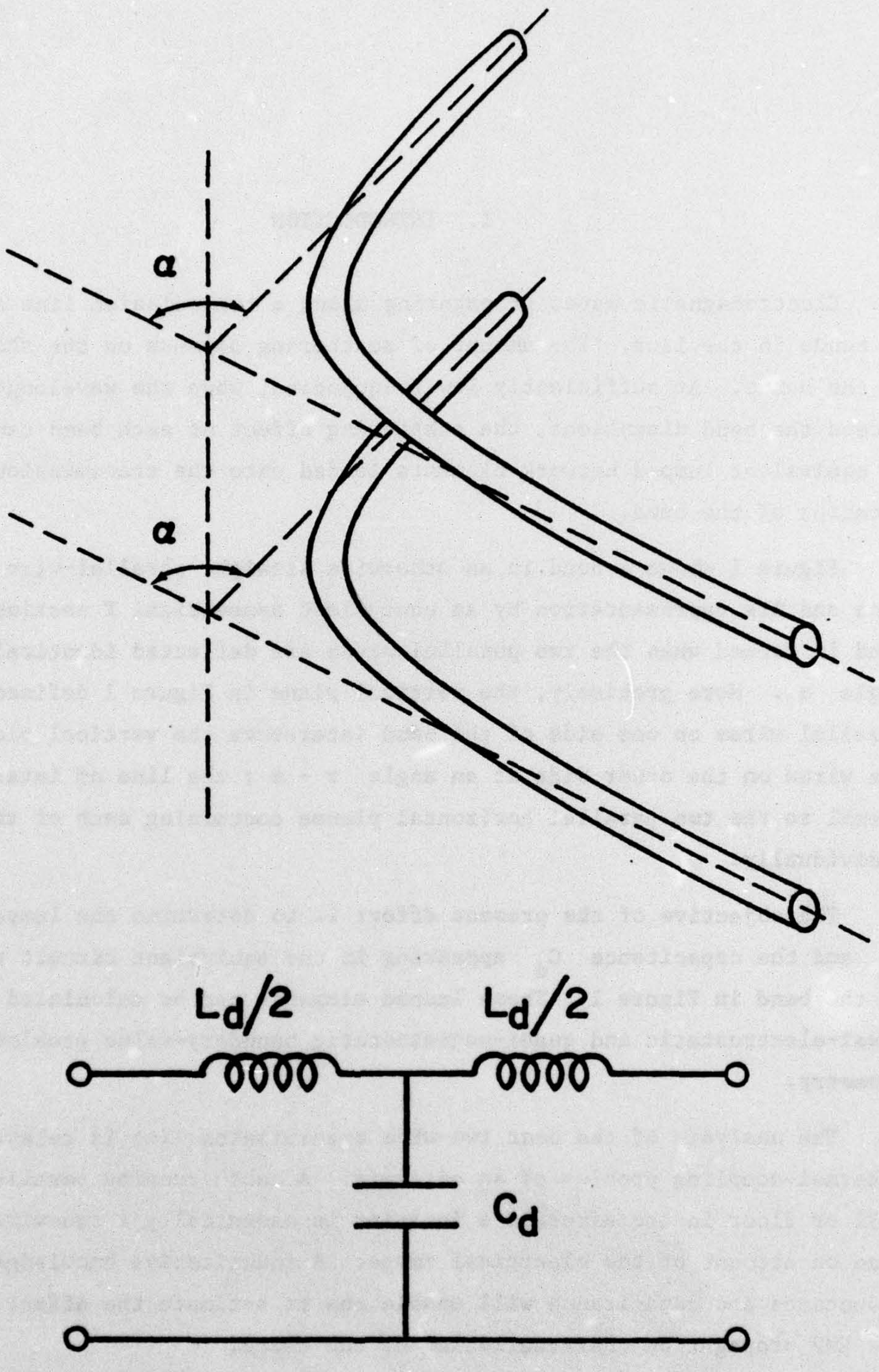

Figure 1. A bend in a parallel-wire transmission line and its equivalent circuit representation.

bend. Specifically, the bend is modeled here by a circular arc with a finite radius. The abrupt bend is recovered in the zero-radius limit. The circular arc is clearly a more realistic model of a cable bend. The calculation shows that the abrupt bend model becomes unreliable when the bend angle α is close to π .

II. GEOMETRY OF THE PROBLEM

Figure 2 shows a model of a bend in an infinitely-long two-wire transmission line. The line consists of two identical parallel conducting cylinders. The radius of each cylinder is a ; the separation of their center lines is $2b$. When the model is applied to the situation of a single conductor over a conducting ground, the parameter b becomes the height of the conductor center line above ground. In the following it will be assumed that the conductors are thin wires so that b is much greater than a .

The bend in each wire is modeled by a circular arc connecting the two semi-infinite straight sections of the wire. The radius of the arc is R ; the angle of the arc is the bend angle α . The two points on the center line at which the circular arc is joined to the straight sections are located at $(x_o , \pm y_o , \pm b)$ with

$$x_o = R \sec\left(\frac{\alpha}{2}\right) - R \cos\left(\frac{\alpha}{2}\right) \tag{1}$$

$$y_o = R \sin\left(\frac{\alpha}{2}\right) \tag{2}$$

The $\pm$ sign of b refers to the upper and lower wires, respectively. In the limit of vanishingly small R , both x_o and y_o tend to zero. The bend geometry degenerates to that of an abrupt V-shaped bend at the coordinate origin.

The total inductance and capacitance of an infinite transmission line are infinite quantities. But the bend inductance L_d and the bend capacitance C_d are finite since they stem from localized deviations in the line geometry. They are functions of the geometrical parameters R , α , a and b .

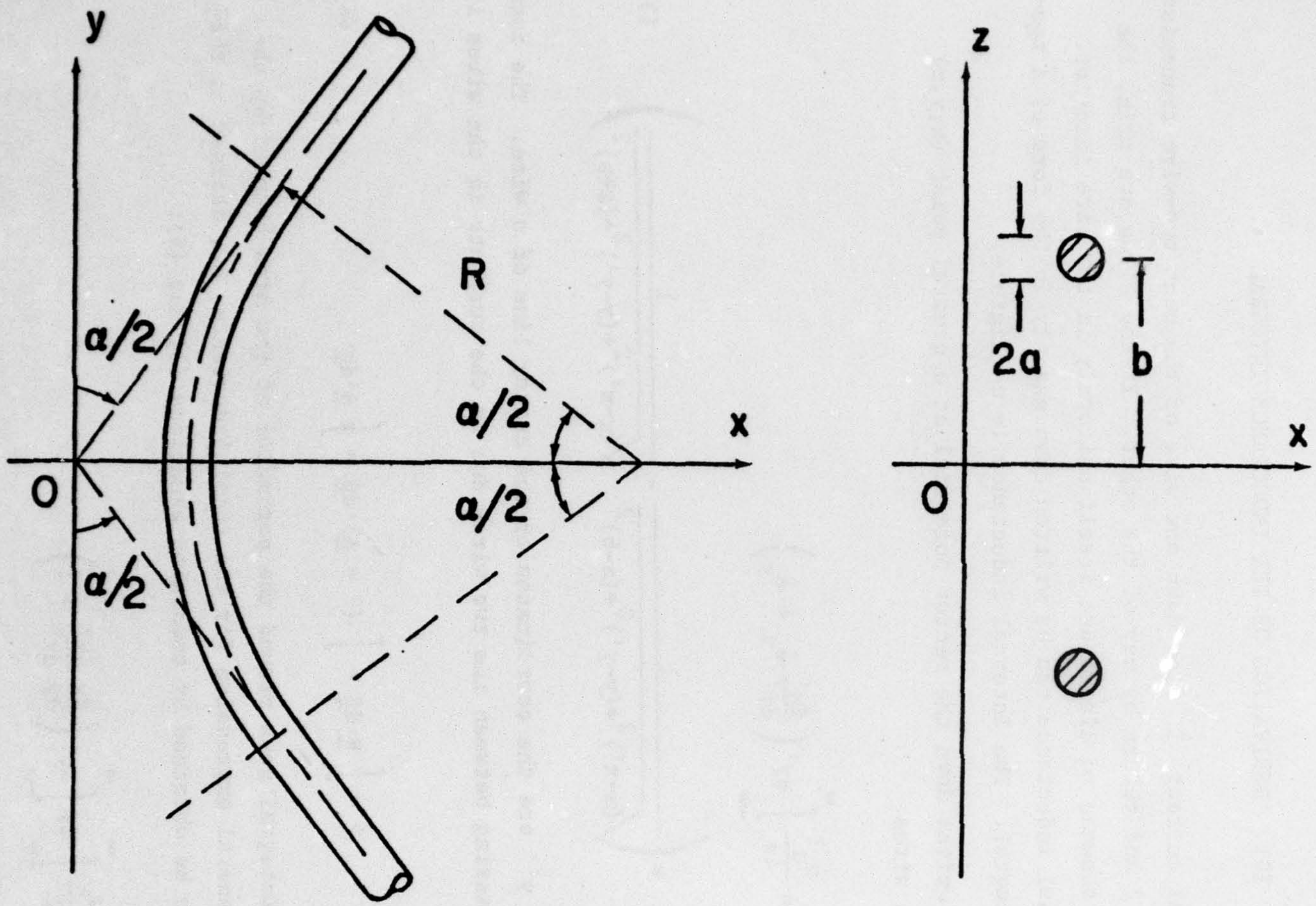

Figure 2. Geometry of a bend in a two-wire transmission line. The bend is modeled by a circular arc of radius R and angle α .

III. DERIVATION OF THE INDUCTANCE INTEGRAL

Let a total current I flow down one wire of the bent two-wire transmission line in Figure 2 and return by way of the other. If the wires are thin, the current can be assumed to distribute itself uniformly in the wire interior. Then the external inductance can be written down exactly in the form of a two-dimensional integral. The internal inductance is negligible.

One first writes down the vector potential at a general point (x,y,z) exterior to the wires

$$\underline{A}(x,y,z) = \frac{\mu_o I}{4\pi} \int_{-\infty}^{\infty} dy' \left(\frac{dx'}{dy'} \hat{e}_x + \hat{e}_y \right) \times \left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-b)^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+b)^2}} \right) \tag{3}$$

where x' and y' are the coordinates of the center line of a wire. The total magnetic flux passing between the two wires due to the currents in the wires is given by

$$\Phi = \int \underline{B} \cdot d\underline{S} = \int (\nabla \times \underline{A}) \cdot d\underline{S} = \oint \underline{A} \cdot d\underline{r} \tag{4}$$

where the line integral goes around the perimeter of the area bounded by the two wires. A general expression for the total inductance L , defined as the ratio Φ/I , can be obtained by combining equations (3) and (4):

$$L = \frac{\mu_o}{2\pi} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dy' \left(\frac{dx}{dy} \frac{dx'}{dy'} + 1 \right) \times \left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+a^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(2b-a)^2}} \right) \tag{5}$$

which is symmetric in the pairs (x,y) and (x',y').

To evaluate the integral (5) one must first supply from the bend geometry the functional relation between x and y, and similarly between x' and y', in the form

$$x = f(y) , \qquad x' = f(y') \tag{6}$$

The relation describes the locus of the center line of a wire. For the case of the circular bend in Figure 2, one has

$$x = f_1(y) = \begin{cases} R \sec\left(\frac{\alpha}{2}\right) - \sqrt{R^2-y^2} & |y| < y_o \\ |y| \tan\left(\frac{\alpha}{2}\right) & |y| > y_o \end{cases} \tag{7}$$

where y_o is defined by (2). For the case of the abrupt bend, one has

$$x = f_2(y) = |y| \tan\left(\frac{\alpha}{2}\right) \tag{8}$$

For the case of the straight transmission line, one has

$$x = f_3(y) = 0 \tag{9}$$

By evaluating the integral (5) with the three different functional expressions f_1, f_2 and f_3 in (7), (8) and (9), one obtains three inductances L_1, L_2 and L_3. They are respectively the total inductance of a line with a circular bend, an abrupt bend, and no bend. All three are linearly divergent quantities. However, the equivalent inductance L_d of the circular bend given by the difference

$$L_d = L_1 - L_3 \tag{10}$$

is finite. In the following two sections the calculation of L_d will proceed in two steps. In Section IV one calculates the inductance difference

$$L_d' = L_2 - L_3 \tag{11}$$

This quantity is the equivalent inductance of an abrupt bend. In Section V one calculates the difference

$$L_d'' = L_1 - L_2 \tag{12}$$

due to the deviation of a circular bend from an abrupt bend. The desired circular bend inductance L_d is then given by the sum

$$L_d = L_d' + L_d'' . \tag{13}$$

IV. INDUCTANCE OF AN ABRUPT CABLE BEND

The equivalent inductance L'_d of an abrupt bend through an angle α in a two-wire transmission line is defined by expression (11). The two inductance integrals L_2 and L_3 are explicitly given by

$$L_2 = \frac{\mu_o}{2\pi} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dy' \left(1 + \lambda^2 \frac{yy'}{|yy'|} \right) \left(\frac{1}{\sqrt{\lambda^2(|y|-|y'|)^2 + (y-y')^2 + a^2}} - \frac{1}{\sqrt{\lambda^2(|y|-|y'|)^2 + (y-y')^2 + (2b-a)^2}} \right) \tag{14}$$

and

$$L_3 = \frac{\mu_o}{2\pi} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dy' \left(\frac{1}{\sqrt{(y-y')^2 + a^2}} - \frac{1}{\sqrt{(y-y')^2 + (2b-a)^2}} \right) \tag{15}$$

where

$$\lambda = \tan\left(\frac{\alpha}{2}\right) \tag{16}$$

One can show that the contribution to L_2 from those regions of integration in which y and y' are of the same sign exactly cancel the corresponding contribution to L_3. The nonzero contribution to L'_d comes from the remaining regions in which y and y' are of opposite signs. In physical terms this result means that the self inductances of the two semi-infinite straight sections of the bent transmission line are unaffected by the bend; the bend inductance L'_d is entirely due to the change in the mutual inductance of the two semi-infinite sections. This change is given by

$$L'_d = F(\lambda) - F(0) \tag{17}$$

where

$$F(\lambda) = \frac{\mu_o}{\pi}(1-\lambda^2) \int_0^\infty dy \int_{-\infty}^0 dy' \left(\frac{1}{\sqrt{\lambda^2(y+y')^2 + (y-y')^2 + a^2}} - \frac{1}{\sqrt{\lambda^2(y+y')^2 + (y-y')^2 + (2b-a)^2}} \right) \tag{18}$$

The integral $F(\lambda)$ can be worked out analytically by the following device. First perform a change of integration variables:

$$u = y - y' , \qquad v = y + y' \tag{19}$$

The corresponding change in the integration is

$$\int_0^\infty dy \int_{-\infty}^0 dy' \rightarrow \frac{1}{2}\int_0^\infty du \int_{-u}^u dv \tag{20}$$

Next make the substitution $v = ut$ and interchange the order of integration. The two ensuing integrations, first over u and then over t, are both elementary. The result is

$$F(\lambda) = \frac{2\mu_o}{\pi}(b-a)\frac{1-\lambda^2}{\lambda}\tan^{-1}\lambda \tag{21}$$

Using definitions (16) and (17), one arrives at the simple explicit result

$$L_d' = \frac{2\mu_o(b-a)}{\pi}(\alpha \cot \alpha - 1) \tag{22}$$

This is the exact formula for the equivalent inductance of an abrupt cable bend within the thin-wire assumption. By contrast, the more complicated expression given by King [3] is only approximate.

A plot of L_d' versus α is shown in Figure 3. One sees that the bend inductance is a negative quantity. Its magnitude increases sharply as the bend approaches a hair-pin bend ($\alpha = \pi$).

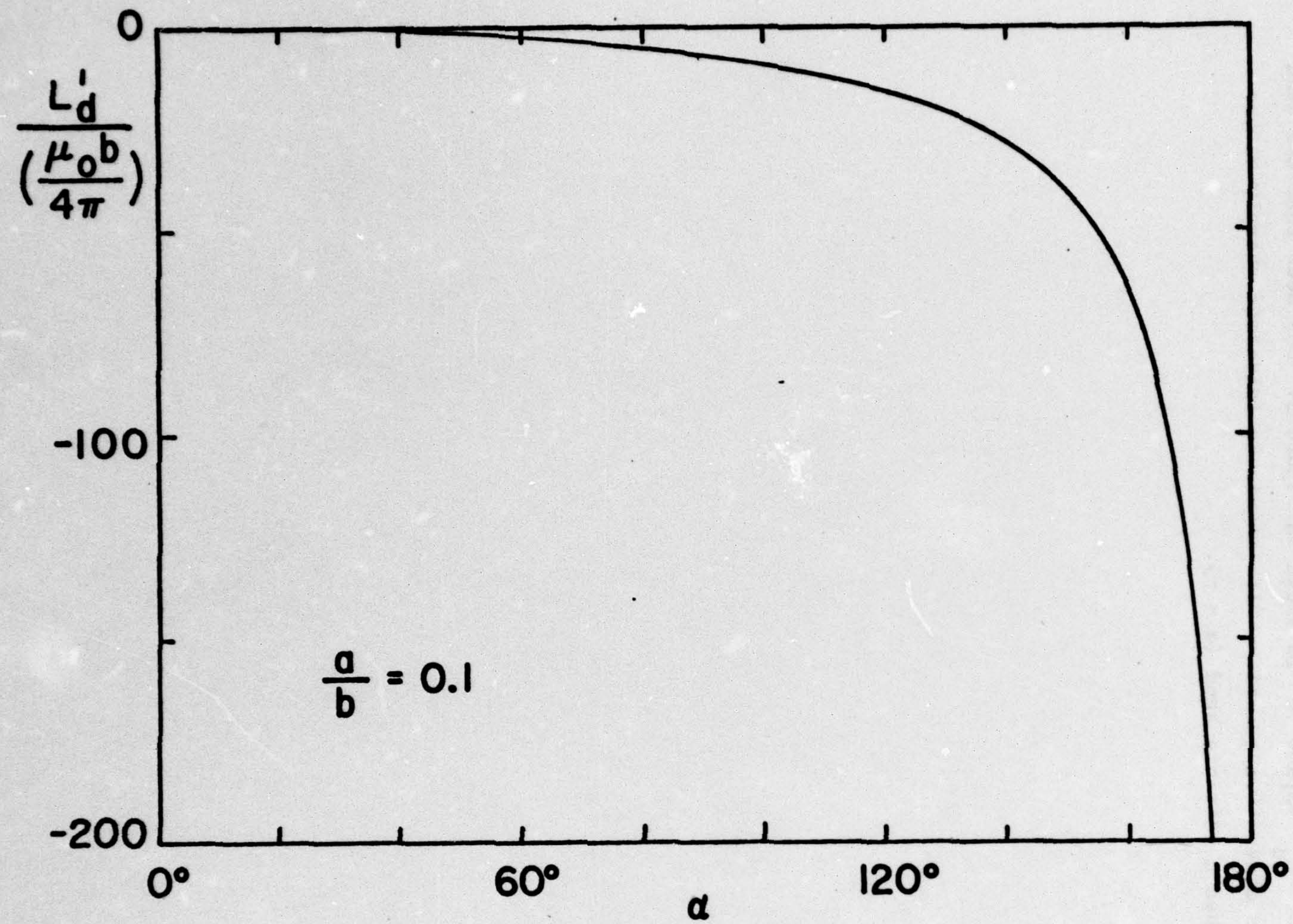

Figure 3. Plot of the equivalent inductance L'_d of an abrupt cable bend versus the bend angle α .

V. INDUCTANCE OF A CIRCULAR CABLE BEND

The change L_d'' in the equivalent inductance when one goes from an abrupt cable bend to a smooth cable bend modeled by a circular arc is given by expression (12). The inductance L_1 is defined by (5) with $x = f_1(y)$ and $x' = f_1(y')$; L_2 is similarly defined with $x = f_2(y)$ and $x' = f_2(y')$. From equations (7) and (8) it is clear that $f_1(y)$ and $f_2(y)$ are identical for $|y| > y_o$. Consequently the nonzero contributions to L_d'' come from regions of integration in which either y or y' or both lie within the interval $(-y_o , y_o)$. On the y-y' plane these regions form a cross, as shown in Figure 4. The contributions to L_d'' consist of a part from the central square of the cross and a part from the four semi-infinite strips making up the four branches:

$$L_d'' = L_d''(\text{square}) + L_d''(\text{strips}) \tag{23}$$

These partial contributions are expressible as

$$\begin{aligned} L_d''(\text{square}) &= G_1 - G_2 \\ L_d''(\text{strips}) &= G_3 - G_4 \end{aligned} \tag{24}$$

where the quantities G_1 , G_2 , G_3 and G_4 are double integrals given explicitly by

$$\begin{aligned} G_1 = \frac{\mu_o}{2\pi}\int_{-y_o}^{y_o} dy \int_{-y_o}^{y_o} dy' \left(1 + \frac{yy'}{\sqrt{(R^2-y^2)(R^2-y'^2)}}\right)\Bigg(& \frac{1}{\sqrt{(\sqrt{R^2-y^2} - \sqrt{R^2-y'^2})^2 + (y-y')^2 + a^2}} \\ & - \frac{1}{\sqrt{(\sqrt{R^2-y^2} - \sqrt{R^2-y'^2})^2 + (y-y')^2 + (2b-a)^2}}\Bigg) \end{aligned} \tag{25}$$

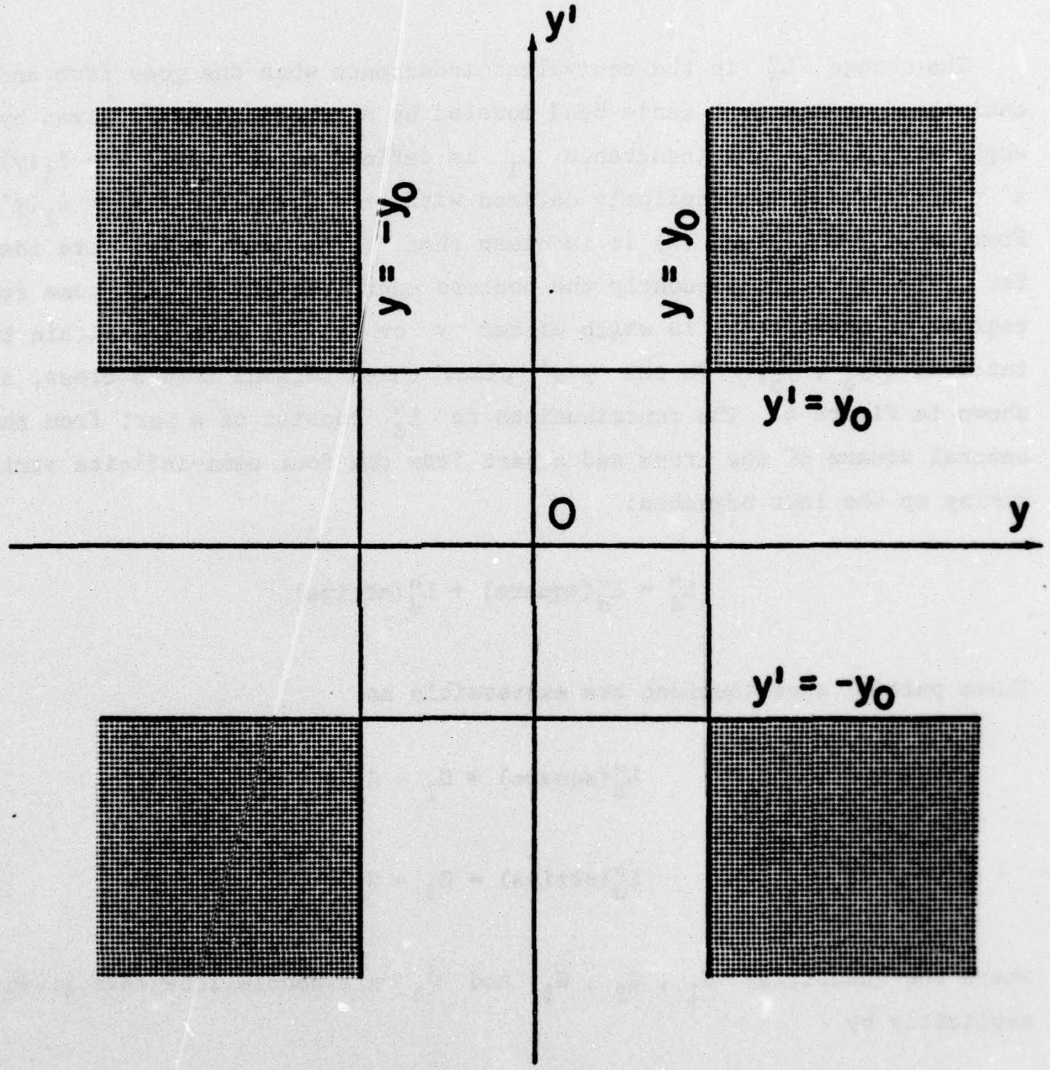

Figure 4. Domain of integration for calculating the inductance difference L_d'' between a circular bend and an abrupt bend. Nonzero contributions to L_d'' come from the undotted cross-shaped region.

$$G_2 = \frac{\mu_o}{2\pi}\int_{-y_o}^{y_o} dy \int_{-y_o}^{y_o} dy' \left(1 + \lambda^2 \frac{yy'}{|yy'|}\right)\left(\frac{1}{\sqrt{\lambda^2(|y|-|y'|)^2 + (y-y')^2 + a^2}}\right.$$

$$\left. - \frac{1}{\sqrt{\lambda^2(|y|-|y'|)^2 + (y-y')^2 + (2b-a)^2}}\right) \tag{26}$$

$$G_3 = \frac{2\mu_o}{\pi}\int_{y_o}^{\infty} dy \int_{-y_o}^{y_o} dy' \left(1 + \frac{\lambda y'}{\sqrt{R^2-y'^2}}\right)\left(\frac{1}{\sqrt{(\lambda y - R\sqrt{1+\lambda^2} + \sqrt{R^2-y'^2})^2 + (y-y')^2 + a^2}}\right.$$

$$\left. - \frac{1}{\sqrt{(\lambda y - R\sqrt{1+\lambda^2} + \sqrt{R^2-y'^2})^2 + (y-y')^2 + (2b-a)^2}}\right) \tag{27}$$

$$G_4 = \frac{2\mu_o}{\pi}\int_{y_o}^{\infty} dy \int_{-y_o}^{y_o} dy' \left(1 + \frac{\lambda^2 y'}{|y'|}\right)\left(\frac{1}{\sqrt{\lambda^2(y-|y'|)^2 + (y-y')^2 + a^2}}\right.$$

$$\left. - \frac{1}{\sqrt{\lambda^2(y-|y'|)^2 + (y-y')^2 + (2b-a)^2}}\right) \tag{28}$$

All four integrals are finite. The contribution L''_d (square) is the shift in the self inductance of the bent section between y_o and $-y_o$ when the bend geometry is changed from an abrupt bend to a smooth bend. The contribution L''_d (strips) is the corresponding shift in the mutual inductance between the bent section and the two adjoining semi-infinite straight sections.

One integration of each of the four double integrals can be carried out. In the case of G_1, it is convenient to first introduce angle variables φ and φ' such that

$$y = R \sin \varphi, \qquad y' = R \sin \varphi' \tag{29}$$

as well as their difference u and sum v :

$$u = \varphi - \varphi' , \qquad v = \varphi + \varphi' \tag{30}$$

The v-integration can at once be performed, with the following result:

$$G_1 = \frac{\mu_o R^2}{\pi} \int_0^{\alpha} du(\alpha - u)\cos u \left(\frac{1}{\sqrt{2R^2 + a^2 - 2R^2 \cos u}} - \frac{1}{\sqrt{2R^2 + (2b-a)^2 - 2R^2 \cos u}} \right) \tag{31}$$

In the case of G_2 , one introduces the difference u and sum v :

$$u = y - y' , \qquad v = y + y' \tag{32}$$

and then integrates over one of them. The result reads

$$G_2 = \frac{2\mu_o}{\pi} (1+\lambda^2) \int_0^{y_o} du(y_o - u) \left(\frac{1}{\sqrt{(1+\lambda)^2 u^2 + a^2}} - \frac{1}{\sqrt{(1+\lambda^2)u^2 + (2b-a)^2}} \right)$$

$$+ \frac{\mu_o}{\pi} (1-\lambda^2) \int_0^{y_o} dv[\Phi(2y_o - v, v) - \Phi(v,v)] \tag{33}$$

where

$$\Phi(u,v) = \ln \left(\frac{u + \sqrt{\lambda^2 v^2 + u^2 + a^2}}{u + \sqrt{\lambda^2 v^2 + u^2 + (2b-a)^2}} \right) \tag{34}$$

In the cases of G_3 and G_4 , one can carry out the y-integration directly without any change of variables, obtaining thereby the expressions

$$G_3 = \frac{2\mu_o}{\pi} \frac{1}{\sqrt{1+\lambda^2}} \int_{-y_o}^{y_o} dy' \left(1 + \frac{\lambda y'}{\sqrt{R^2-y'^2}}\right) \Psi(R\sqrt{1+\lambda^2} - \sqrt{R^2-y'^2}\ ,\ y') \tag{35}$$

$$G_4 = \frac{2\mu_o}{\pi} \frac{1}{\sqrt{1+\lambda^2}} \int_{-y_o}^{y_o} dy' \left(1 + \frac{\lambda^2 y'}{|y'|}\right) \Psi(\lambda|y'|\ ,\ y') \tag{36}$$

where

$$\Psi(x,y) = \ln\left(\frac{\sqrt{1+\lambda^2}\ \sqrt{(\lambda y_o-x)^2 + (y_o-y)^2 + (2b-a)^2} + (1+\lambda^2)y_o - \lambda x - y}{\sqrt{1+\lambda^2}\ \sqrt{(\lambda y_o-x)^2 + (y_o-y)^2 + a^2} + (1+\lambda^2)y_o - \lambda x - y}\right) \tag{37}$$

The remaining integration in G_2 and G_4 can be performed exactly. However, the explicit integration produces such a proliferation of terms that the results are practically useless. The integrals G_1 and G_3 in (31) and (35) contain parts which are essentially integrals of incomplete elliptic integrals. They are beyond the limits of the art of analytical integration. One must ultimately resort to numerical integration for their evaluation. For numerical purposes, it is more advantageous to retain all four integrals in the forms (31), (33), (35) and (36).

The equivalent inductance L_d of a smooth bend consisting of a circular arc of radius R and angle α is therefore given by

$$L_d = L_d' + G_1 - G_2 + G_3 - G_4 \tag{38}$$

where L_d' is the equivalent inductance of an abrupt bend of angle α as given in (22). The four G integrals are evaluated numerically for the cases $b = 10a$ and $R = 2b$ and $4b$. The values of L_d are plotted versus α in Figure 5. The inductance of an abrupt bend ($R = 0$) is also shown for comparison. It is seen that the dependence of L_d on the bend radius R is very pronounced

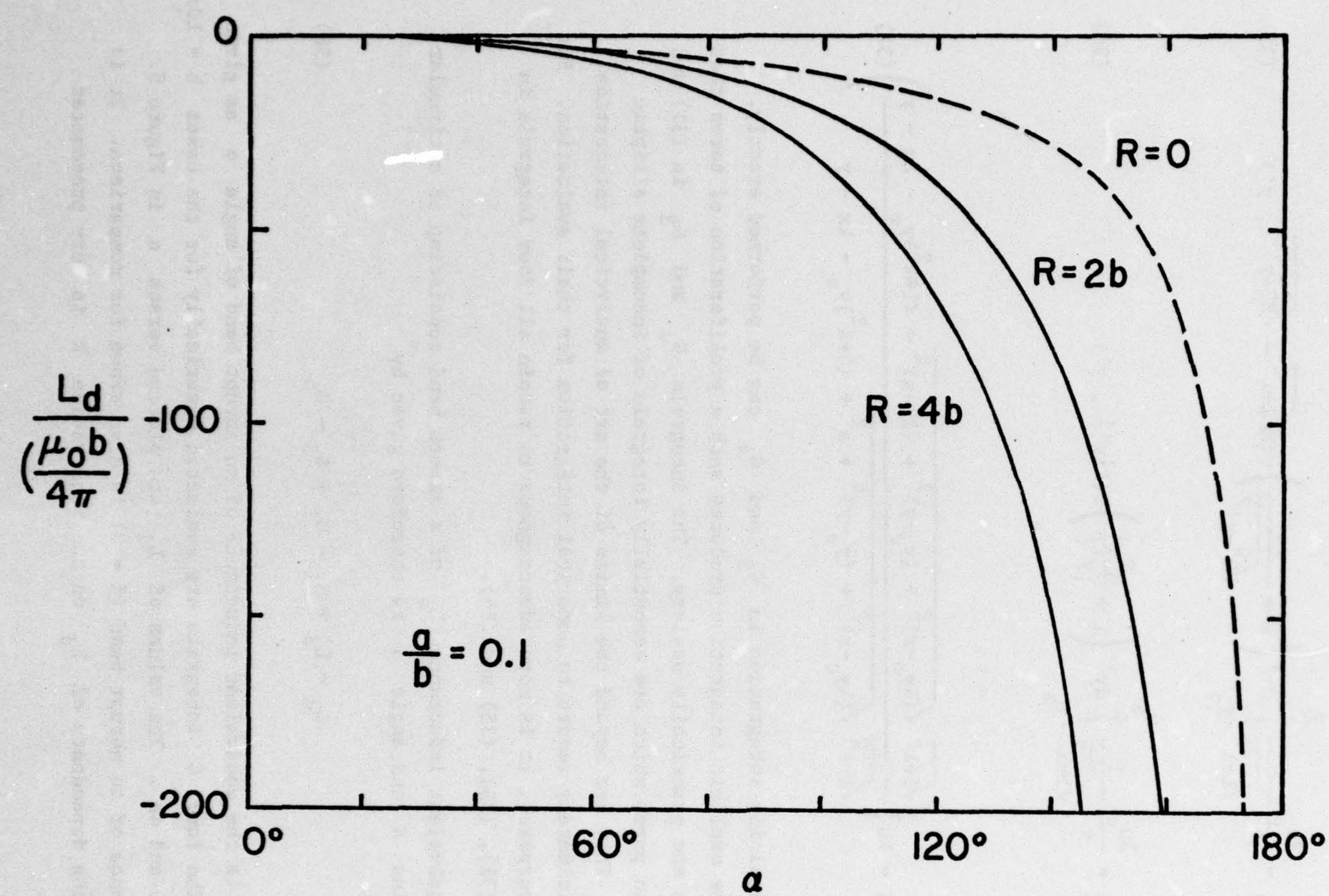

Figure 5. Plot of the equivalent inductance L_d of a circular cable bend of radius R versus the bend angle α. The broken line is the inductance L_d' of an abrupt bend $(R = 0)$.

for large bend angle α . At the chosen values of the parameters the abrupt bend can be said to approximate the smooth bend only for α less than about 40°.

VI. INTEGRAL-EQUATION FORMULATION OF THE CAPACITANCE PROBLEM

Consider the bent two-wire transmission line in the coordinate system in Figure 2. Let the upper wire at $z = b$ be charged to a potential $V_o/2$, and the lower wire at $z = -b$ to a potential $-V_o/2$. The potential difference between the two wires is therefore V_o. The charge densities per unit length on the upper and lower wires will be denoted by $\pm\sigma$, respectively. It will be assumed that the two wires are thin, so that the wire radius is much smaller than the wire separation. One can then consider the charges on the wires as being concentrated on the center lines of the wires; and σ can be expressed as a function of the y-coordinate alone.

At a general point (x,y,z) exterior to the two wires the total electrostatic potential due to the wires is obtained by summing up the contributions from all the charge elements along the center lines:

$$V(x,y,z) = \frac{1}{4\pi\varepsilon_o} \int_{-\infty}^{\infty} dy' \frac{ds'}{dy'} \sigma(y') \left(\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-b)^2}} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+b)^2}} \right) \tag{39}$$

In the formula s' is the arc length measured along the center line of a wire, so that

$$\frac{ds'}{dy'} = \sqrt{1 + \left(\frac{dx'}{dy'}\right)^2} \tag{40}$$

Strictly speaking, the charge density $\sigma(y')$ is to be determined by requiring that V reduce to $\pm V_o/2$ on the entire surfaces of the upper and lower wires, respectively. However, to be consistent with the thin-wire assumption inherent in formula (39), one can apply the boundary condition only along a line on the surface of each wire. This line will be chosen to be at $z = b - a$ on the upper wire, and at $z = -(b - a)$ on the lower wire. An integral equation for $\sigma(y')$ results:

$$V_o = \frac{1}{2\pi\varepsilon_o} \int_{-\infty}^{\infty} dy' \frac{ds'}{dy'} \sigma(y') \left(\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + a^2}} \right.$$

$$\left. - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (2b-a)^2}} \right) \qquad -\infty < y < \infty \qquad (41)$$

The integral equation (41) is to be solved under a specified functional relation between x and y, and similarly between x' and y'. This relation takes the form

$$x = f(y) , \qquad x' = f(y') \qquad (42)$$

and describes the center line of a wire. Three forms of $f(y)$ will be considered in the following sections. For the line with a circular bend shown in Figure 2, one has

$$x = f_1(y) = \begin{cases} R \sec\left(\frac{\alpha}{2}\right) - \sqrt{R^2-y^2} & |y| < y_o \\ |y| \tan\left(\frac{\alpha}{2}\right) & |y| > y_o \end{cases} \qquad (43)$$

where

$$y_o = R \sin\left(\frac{\alpha}{2}\right) \qquad (44)$$

One also considers the abrupt V-shaped bend obtained in the limit as R tends to zero. The functional relation for this case is

$$x = f_2(y) = |y| \tan\left(\frac{\alpha}{2}\right) \tag{45}$$

Finally, for a straight uniform line with no bend, one has

$$x = f_3(y) = 0 \tag{46}$$

VII. VARIATIONAL REPRESENTATION OF THE CAPACITANCE

It is possible to calculate the capacitance of the two-wire transmission line without solving explicitly the integral equation (41). The way to go about this is to phrase the capacitance calculation as an eigenvalue problem. From well-known results in the calculus of variations, a variational principle for the eigenvalue can be established. It can be applied to obtain an estimate of the capacitance with a judicious choice of the trial function.

Let Q denote the total charge on the upper wire. It is an infinite quantity and can be expressed as a line integral of the line charge density σ :

$$Q = \int_{-\infty}^{\infty} dy' \, \frac{ds'}{dy'} \, \sigma(y') \tag{47}$$

Q is directly proportional to the potential difference V_o , the constant of proportionality being the capacitance C :

$$Q = CV_o \tag{48}$$

Using equations (47) and (48), one can eliminate V_o from the integral equation (41), which then becomes

$$\frac{1}{C}\int_{-\infty}^{\infty} dy' \, \frac{ds'}{dy'} \, \sigma(y') = \int_{-\infty}^{\infty} dy' \, \frac{ds'}{dy'} \, \sigma(y') K(y,y') \qquad -\infty < y < \infty \tag{49}$$

The kernel $K(y,y')$ is defined by

$$K(y,y') = \frac{1}{2\pi\varepsilon_o}\left(\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + a^2}} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (2b-a)^2}}\right) \tag{50}$$

where x and x' are related to y and y' through equation (42).

Equation (49) is linear and homogeneous, with the reciprocal of the capacitance $1/C$ playing the role of an eigenvalue. A variational representation of the eigenvalue for an integral equation of the type (49) is well known [4] and takes the form

$$\frac{1}{C} = \frac{\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dy' \frac{ds}{dy} \frac{ds'}{dy'} \sigma(y) K(y,y') \sigma(y')}{\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dy' \frac{ds}{dy} \frac{ds'}{dy'} \sigma(y) \sigma(y')} \tag{51}$$

By "variational representation" is meant that $1/C$ is a functional on the space of trial charge density functions σ, and that its value attains an absolute minimum at the exact solution of (49). This minimum corresponds to the exact value of the capacitance. If a trial charge density function differing from the exact solution by a small amount $\delta\sigma$ is inserted in (51), the error incurred in the approximate value of the capacitance so obtained is only of order $(\delta\sigma)^2$. Consequently an evaluation of expression (51), even with a very crude trial function, can yield a good estimate of the capacitance.

On physical grounds the charge density per unit length of the two-wire transmission line is uniform except in the vicinity of the bend. For a very long line with a length greatly exceeding the bend dimensions a good trial function is therefore

$$\sigma(y) = \text{constant} \tag{52}$$

With this simple choice expression (51) becomes

$$\frac{1}{C} = \frac{\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dy' \frac{ds}{dy} \frac{ds'}{dy'} K(y,y')}{\int_{-\infty}^{\infty} dy \frac{ds}{dy} \int_{-\infty}^{\infty} dy' \frac{ds'}{dy'}} \tag{53}$$

which depends only on the geometry of the bend. By applying to the kernel $K(y,y')$ in (50) the three functional relations f_1, f_2 and f_3 defined in equations (43), (45) and (46), one obtains three kernels K_1, K_2 and K_3. These, when inserted into formula (53), generate three capacitances C_1, C_2 and C_3. They are, respectively, the total capacitance of a two-wire transmission line with a circular bend, an abrupt bend and no bend.

In the following two sections the equivalent capacitances of an abrupt bend and a circular bend are calculated from formula (53). In Section VIII the equivalent capacitance of an abrupt bend, denoted by C'_d and defined as the difference

$$C'_d = C_2 - C_3 \tag{54}$$

is evaluated in closed form. In Section IX the difference C''_d between the circular bend capacitance and the abrupt bend capacitance, given by

$$C''_d = C_1 - C_2 \tag{55}$$

is expressed in the form of one-dimensional integrals ready for computation. The equivalent capacitance C_d of the circular bend is then obtained as

$$C_d = C_1 - C_3 = C'_d + C''_d \tag{56}$$

VIII. CAPACITANCE OF AN ABRUPT CABLE BEND

Consider a two-wire transmission line with an abrupt bend through an angle α. This bend can be regarded as the limit of the circular bend in Figure 2 when the bend radius R tends to zero. Let the transmission line be of finite length initially and stretch from $y = D$ to $y = -D$. Eventually the constant D will be allowed to tend to infinity. The total capacitance C_2 of this line is calculable from formula (53):

$$\frac{1}{C_2} = \frac{1}{S_2^2} \int_{-D}^{D} dy \int_{-D}^{D} dy' \, \frac{ds_2}{dy} \frac{ds_2'}{dy'} K_2(y,y') \tag{57}$$

The subscript 2 will everywhere refer to the line with an abrupt bend. The kernel K_2 is given by

$$K_2(y,y') = \frac{1}{2\pi\varepsilon_o}\left(\frac{1}{\sqrt{\lambda^2(|y|-|y'|)^2 + (y-y')^2 + a^2}} - \frac{1}{\sqrt{\lambda^2(|y|-|y'|)^2 + (y-y')^2 + (2b-a)^2}}\right) \tag{58}$$

with

$$\lambda = \tan\left(\frac{\alpha}{2}\right) \tag{59}$$

S_2 is the total length of the line between $y = D$ and $y = -D$:

$$S_2 = \int_{-D}^{D} dy' \, \frac{ds_2'}{dy'} = 2D\sqrt{1 + \lambda^2} \tag{60}$$

since, by (40) and (45),

$$\frac{ds_2'}{dy'} = \sqrt{1 + \lambda^2} \tag{61}$$

Now consider a straight two-wire transmission line of the same total length. That is, its length S_3 is given by

$$S_3 = S_2 = 2D\sqrt{1+\lambda^2} \tag{62}$$

where the subscript 3 refers to the straight transmission line. This line can be taken to lie parallel to the y-axis in Figure 2, and to stretch between $y = D\sqrt{1+\lambda^2}$ and $y = -D\sqrt{1+\lambda^2}$. According to formula (53), its total capacitance C_3 is

$$\frac{1}{C_3} = \frac{1}{S_3^2}\int_{-D\sqrt{1+\lambda^2}}^{D\sqrt{1+\lambda^2}} dy \int_{-D\sqrt{1+\lambda^2}}^{D\sqrt{1+\lambda^2}} dy'\, K_3(y,y') \tag{63}$$

where

$$K_3(y,y') = \frac{1}{2\pi\epsilon_o}\left(\frac{1}{\sqrt{(y-y')^2 + a^2}} - \frac{1}{\sqrt{(y-y')^2 + (2b-a)^2}}\right) \tag{64}$$

A simple change of variables reduces (63) to the form

$$\frac{1}{C_3} = \frac{1+\lambda^2}{S_3^2}\int_{-D}^{D} dy \int_{-D}^{D} dy'\, K_3\left(\sqrt{1+\lambda^2}\; y\; ,\; \sqrt{1+\lambda^2}\; y'\right) \tag{65}$$

Substituting formulas (57) and (65) into the identity

$$C_2 - C_3 = C_2C_3\left(\frac{1}{C_3} - \frac{1}{C_2}\right) \tag{66}$$

and making use of relations (61) and (62), one obtains an expression for the equivalent capacitance C_d' of an abrupt cable bend defined in (54):

$$C_d' = \frac{C_2 C_3 (1+\lambda^2)}{S_3^2} \int_{-D}^{D} dy \int_{-D}^{D} dy' \left[K_3(\sqrt{1+\lambda^2}\ y\ ,\ \sqrt{1+\lambda^2}\ y') - K_2(y,y') \right] \tag{67}$$

When the total length of the line is allowed to increase, the effect of the bend on the total capacitance of the line becomes negligible. The total capacitance approaches the product of the line length and the constant capacitance per unit length of the uniform line. In mathematical terms, one has

$$\lim_{D \to \infty} C_2 = C_3 = \kappa S_3 \tag{68}$$

where

$$\kappa = \frac{\pi \varepsilon_o}{\ell n \left(\frac{2b}{a}\right)} \qquad b >> a \tag{69}$$

is the well-known capacitance per unit length of the uniform two-wire transmission line. Therefore in the limit $D \to \infty$, formula (67) goes over to the desired expression for the equivalent capacitance of an abrupt bend in an infinite two-wire transmission line:

$$C_d' = \kappa^2 (1+\lambda^2) \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dy' \left[K_3(\sqrt{1+\lambda^2}\ y\ ,\ \sqrt{1+\lambda^2}\ y') - K_2(y,y') \right] \tag{70}$$

It is easy to see from (58) and (64) that the integrand in (70) vanishes identically whenever y and y' are of the same sign. The nonzero contributions to C_d' can be rearranged as follows:

$$C_d' = \tilde{F}(0) - \tilde{F}(\lambda) \tag{71}$$

where

$$\tilde{F}(\lambda) = \frac{\kappa^2(1+\lambda^2)}{\pi\varepsilon_o}\int_0^\infty dy \int_{-\infty}^0 dy' \left(\frac{1}{\sqrt{\lambda^2(y+y')^2 + (y-y')^2 + a^2}} - \frac{1}{\sqrt{\lambda^2(y+y')^2 + (y-y')^2 + (2b-a)^2}} \right) \tag{72}$$

The integral is the same as that appearing in expression (18) of Section IV, and can be evaluated analytically by the method described therein. The result is

$$\tilde{F}(\lambda) = \frac{2\kappa^2(b-a)}{\pi\varepsilon_o}\,\frac{1+\lambda^2}{\lambda}\,\tan^{-1}\lambda \tag{73}$$

Combining expressions (59), (69), (71) and (73), one obtains the equivalent capacitance of an abrupt bend in the simple formula

$$C_d' = \frac{2\pi\varepsilon_o(b-a)}{\left[\ln\left(\frac{2b}{a}\right)\right]^2}(1 - \alpha \csc \alpha) \tag{74}$$

For completeness one quotes here the associated equivalent inductance L_d' of the abrupt bend derived in equation (22) of Section IV:

$$L_d' = \frac{2\mu_o(b-a)}{\pi}(\alpha \cot \alpha - 1) \tag{75}$$

The expression (74) for the bend capacitance C_d' is evaluated for the case $b = 10a$, and plotted versus the bend angle α in Figure 6. The equivalent capacitance of an abrupt cable bend has previously been calculated by Tomiyasu [1] and King [5] using a different approximate method. Expression (74) agrees with their result to within a few percent. It is, however, simpler in form and applies to a wider range of the bend angle α.

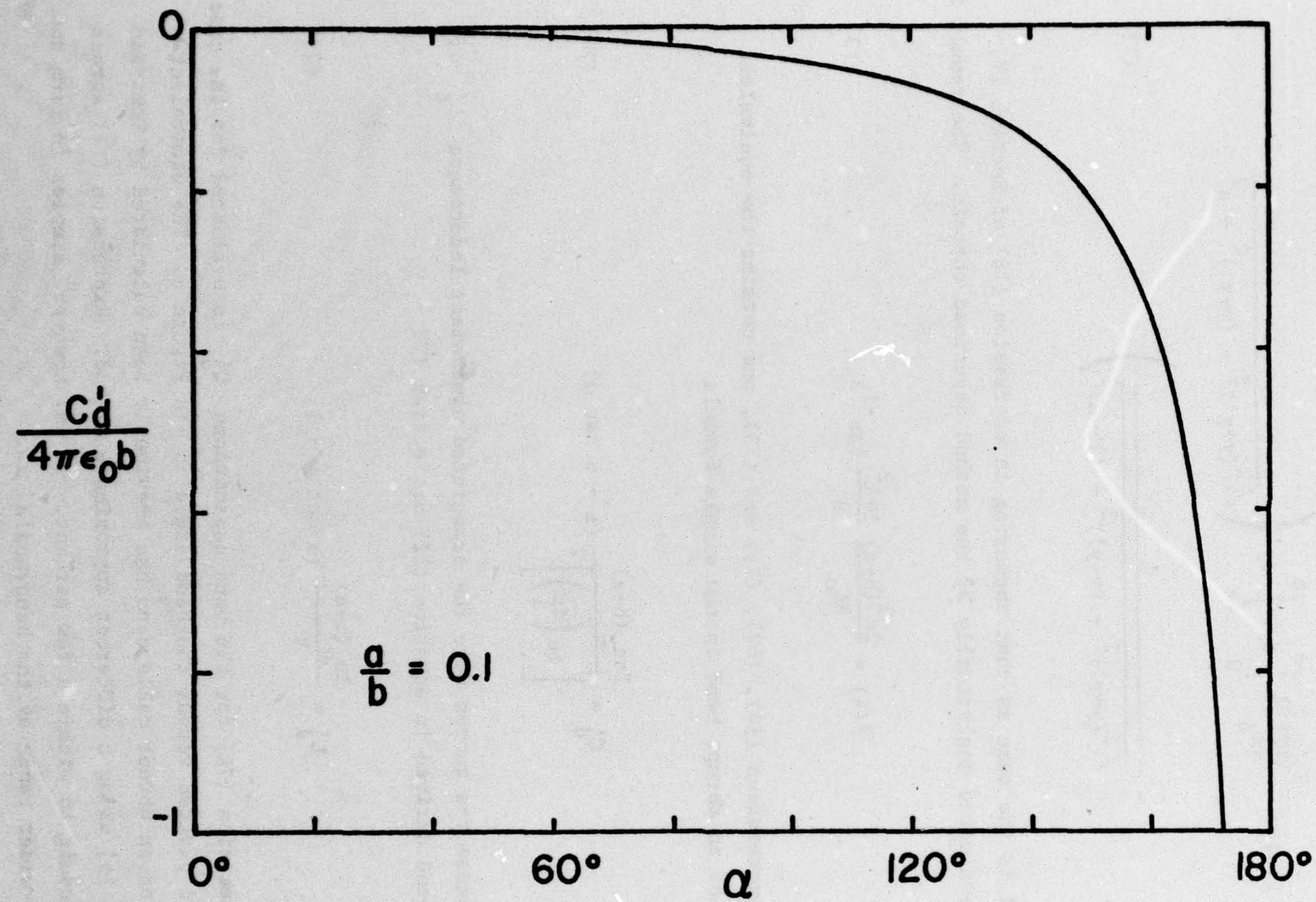

Figure 6. Plot of the equivalent capacitance C_d' of an abrupt cable bend versus the bend angle α .

IX. CAPACITANCE OF A CIRCULAR CABLE BEND

The equivalent capacitance C_d of a circular cable bend is the sum of the equivalent capacitance C'_d of an abrupt bend calculated in Section VIII and the correction term C''_d defined in equation (55). The evaluation of C''_d is undertaken in this section.

Applying formula (53) successively to the two transmission lines with the circular and abrupt bends and taking the difference of the two resulting expressions, one obtains

$$\frac{S_2^2}{C_2} - \frac{S_1^2}{C_1} = W \tag{76}$$

On the left-hand side, S_1 and S_2 are the total lengths of the two transmission lines:

$$S_1 = \int_{-\infty}^{\infty} dy' \frac{ds'_1}{dy'} , \qquad S_2 = \int_{-\infty}^{\infty} dy' \frac{ds'_2}{dy'} \tag{77}$$

They are both linearly divergent quantities. The right-hand side of (76) is a two-dimensional integral:

$$W = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dy' \left[\frac{ds_2}{dy} \frac{ds'_2}{dy'} K_2(y,y') - \frac{ds_1}{dy} \frac{ds'_1}{dy'} K_1(y,y') \right] \tag{78}$$

The kernel $K_2(y,y')$ has been written out explicitly in equation (58). The kernel $K_1(y,y')$ is given by

$$K_1(y,y') = \frac{1}{2\pi\varepsilon_o} \left(\frac{1}{\sqrt{[f_1(y) - f_1(y')]^2 + (y-y')^2 + a^2}} \right.$$

$$- \frac{1}{\sqrt{[f_1(y) - f_1(y')]^2 + (y-y')^2 + (2b-a)^2}} \Bigg) \qquad (79)$$

with $f_1(y)$ defined by (43). The two quantities s_1 and s_2 in (78) are arc lengths along the transmission lines. By (40) one obtains

$$\frac{ds_1'}{dy'} = \sqrt{1 + [f_1'(y')]^2} \ , \qquad \frac{ds_2'}{dy'} = \sqrt{1 + \lambda^2} \qquad (80)$$

The left-hand side of (76) is the difference of two infinite quantities. Its value is nevertheless finite, and is related to the capacitance difference $C_d'' = C_1 - C_2$. Introducing the relations

$$C_2 = C_1 - C_d'' \ , \qquad S_2 = S_1 + \Delta S \qquad (81)$$

where C_d'' and ΔS are finite and C_1 and S_1 are infinite, one finds that

$$\frac{S_2^2}{C_2} - \frac{S_1^2}{C_1} = \left(\frac{S_1}{C_1}\right)^2 C_d'' + 2\left(\frac{S_1}{C_1}\right)\Delta S \qquad (82)$$

The ratio C_1/S_1 in (82) is simply the capacitance per unit length of the infinite bent transmission line. It is equal to κ defined in (69) for the uniform line. Furthermore ΔS is the difference in the arc length between an abrupt bend and a circular bend. From Figure 2 one can immediately write down

$$\Delta S = R(2\lambda - \alpha) \qquad (83)$$

Substituting (82) and (83) into (76), one finally obtains an expression for the capacitance correction term:

$$C_d'' = \kappa^2 W - 2\kappa R(2\lambda - \alpha) \qquad (84)$$

It remains to evaluate the two-dimensional integral W appearing in (84) and defined in (78). One easily sees from the definition of $f_1(y)$ in (43) that the integrand in (78) vanishes identically whenever both y and y' lie outside the interval $(-y_o, y_o)$. This result is a reflection of the fact that the two differently-bent transmission lines coincide outside the bent sections. Consequently the nonzero contributions to the integral W come only from a certain cross-shaped region on the y-y' plane, as shown in Figure 4. The total contributions consist of a part from the central square of the cross, and a part from the four semi-infinite strips forming the four limbs:

$$W = W(\text{square}) + W(\text{strips}) \tag{85}$$

Each of the two parts can be further subdivided as follows:

$$\begin{aligned} W(\text{square}) &= \tilde{G}_1 - \tilde{G}_2 \\ W(\text{strips}) &= \tilde{G}_3 - \tilde{G}_4 \end{aligned} \tag{86}$$

The four $\tilde{G}$'s are two-dimensional integrals defined explicitly as follows:

$$\tilde{G}_1 = \frac{1+\lambda^2}{2\pi\varepsilon_o}\int_{-y_o}^{y_o} dy \int_{-y_o}^{y_o} dy' \left(\frac{1}{\sqrt{\lambda^2(|y|-|y'|)^2 + (y-y')^2 + a^2}} - \frac{1}{\sqrt{\lambda^2(|y|-|y'|)^2 + (y-y')^2 + (2b-a)^2}}\right) \tag{87}$$

$$\tilde{G}_2 = \frac{R^2}{2\pi\varepsilon_o}\int_{-\alpha/2}^{\alpha/2} d\varphi \int_{-\alpha/2}^{\alpha/2} d\varphi' \left(\frac{1}{\sqrt{2R^2 + a^2 - 2R^2\cos(\varphi-\varphi')}} - \frac{1}{\sqrt{2R^2 + (2b-a)^2 - 2R^2\cos(\varphi-\varphi')}}\right) \tag{88}$$

$$\tilde{G}_3 = \frac{2(1+\lambda^2)}{\pi\varepsilon_o} \int_{y_o}^{\infty} dy \int_{-y_o}^{y_o} dy' \left(\frac{1}{\sqrt{\lambda^2 (y-|y'|)^2 + (y-y')^2 + a^2}} \right.$$

$$\left. - \frac{1}{\sqrt{\lambda^2 (y-|y'|)^2 + (y-y')^2 + (2b-a)^2}} \right) \tag{89}$$

$$\tilde{G}_4 = \frac{2R\sqrt{1+\lambda^2}}{\pi\varepsilon_o} \int_{y_o}^{\infty} dy \int_{-\alpha/2}^{\alpha/2} d\varphi' \left(\frac{1}{\sqrt{(\lambda y - R\sqrt{1+\lambda^2} + R\cos\varphi')^2 + (y - R\sin\varphi')^2 + a^2}} \right.$$

$$\left. - \frac{1}{\sqrt{(\lambda y - R\sqrt{1+\lambda^2} + R\cos\varphi')^2 + (y - R\sin\varphi')^2 + (2b-a)^2}} \right) \tag{90}$$

These four integrals are similar to those encountered in the inductance calculation. Using the method outlined in Section V already, one can reduce the $\tilde{G}$'s to one-dimensional integrals. The result of the reduction is

$$\tilde{G}_1 = \frac{2(1+\lambda^2)}{\pi\varepsilon_o} \int_0^{y_o} du\,(y_o - u) \left(\frac{1}{\sqrt{(1+\lambda^2)u^2 + a^2}} - \frac{1}{\sqrt{(1+\lambda^2)u^2 + (2b-a)^2}} \right)$$

$$+ \frac{1+\lambda^2}{\pi\varepsilon_o} \int_0^{y_o} dv [\Phi(2y_o - v, v) - \Phi(v, v)] \tag{91}$$

$$\tilde{G}_2 = \frac{R^2}{\pi\varepsilon_o} \int_0^{\alpha} du\,(\alpha - u) \left(\frac{1}{\sqrt{2R^2 + a^2 - 2R^2 \cos u}} \right.$$

$$\left. - \frac{1}{\sqrt{2R^2 + (2b-a)^2 - 2R^2 \cos u}} \right) \tag{92}$$

$$\tilde{G}_3 = \frac{2\sqrt{1+\lambda^2}}{\pi\varepsilon_o} \int_{-y_o}^{y_o} dy' \Psi(\lambda|y'|, y') \tag{93}$$

$$\tilde{G}_4 = \frac{2R}{\pi\varepsilon_o} \int_{-\alpha/2}^{\alpha/2} d\varphi' \Psi(R\sqrt{1+\lambda^2} - R\cos\varphi' , R\sin\varphi') \tag{94}$$

with

$$\Phi(u,v) = \ln\left(\frac{u + \sqrt{\lambda^2 v^2 + u^2 + a^2}}{u + \sqrt{\lambda^2 v^2 + u^2 + (2b-a)^2}}\right) \tag{95}$$

and

$$\Psi(x,y) = \ln\left(\frac{\sqrt{1+\lambda^2}\sqrt{(\lambda y_o - x)^2 + (y_o - y)^2 + (2b-a)^2} + (1+\lambda^2)y_o - \lambda x - y}{\sqrt{1+\lambda^2}\sqrt{(\lambda y_o - x)^2 + (y_o - y)^2 + a^2} + (1+\lambda^2)y_o - \lambda x - y}\right) \tag{96}$$

For numerical purposes there is no advantage in trying to further reduce the integrals. They will therefore be left in the present form.

In summary the equivalent capacitance C_d of a circular cable bend of radius R and angle α is given by

$$C_d = C_d' - 2\kappa R(2\lambda - \alpha) + \kappa^2(\tilde{G}_1 - \tilde{G}_2 + \tilde{G}_3 - \tilde{G}_4) \tag{97}$$

The $\tilde{G}$'s are evaluated numerically for the typical cases $b = 10a$ and $R = 2b$ and $4b$. The values of C_d are plotted versus α in Figure 7. The capacitance C_d' of an abrupt bend ($R = 0$) is also shown for comparison. It is obvious from the figure that the abrupt bend is not a good approximation to the smooth bend. The same conclusion was drawn from inductance consideration.

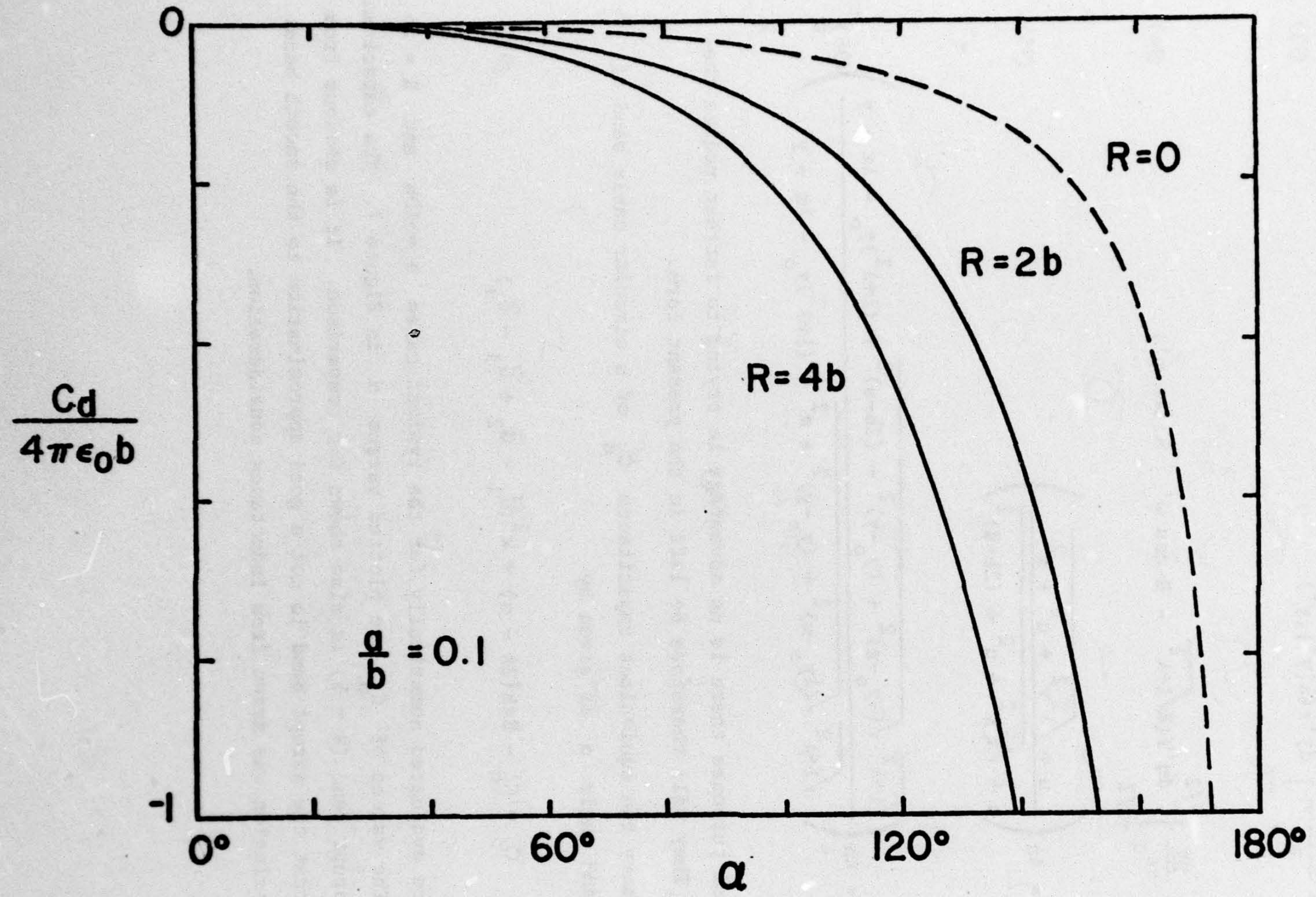

Figure 7. Plot of the equivalent capacitance C_d of a circular cable bend of radius R versus the bend angle α. The broken line is the capacitance C'_d of an abrupt bend $(R = 0)$.

X. CONCLUSION

A symmetrical bend in a two-wire transmission line is modeled by a circular arc of radius R and angle α, as shown in Figures 1 and 2. In the limit as R tends to zero, one obtains the geometry of an abrupt bend. The equivalent bend inductance L_d and the equivalent bend capacitance C_d are functions of four geometrical parameters R, α, a and b, where a is the wire radius and b is one-half the wire separation.

The inductance L_d is calculated exactly within the thin-wire assumption. The capacitance C_d is calculated from a variational principle. Numerical studies lead to the conclusion that both L_d and C_d depend strongly on the bend radius R, so that the abrupt bend is not often a realistic model of a cable bend.

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